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Equisingular Stratifications Associated to Families of Planar Ideals

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1. INTRODUCTION

In Algebraic Geometry it is common to consider families of objects, parametrized by a variety or scheme, say T. Often it is natural to regard two objects X_t and $X_{t'}$ (corresponding to values t, t' of the parameter) as being "equivalent," say because they share some relevant features, e.g., certain important numerical (or geometric, or topological) invariants are the same for both of them. In general this leads to a natural equivalence relation on T (where t and t' in T are related if the corresponding objects are "equivalent"), hence to a partition, or stratification, of the parameter space. What kind of sets are these strata? In general, the best that can be expected is that they be locally closed subsets of (the underlying topological space) of T; hence they will inherit a structure of subvariety, or subscheme of T.

In this paper we give a criterion in order that the members of stratification of a Zariski topological space T will be locally closed sets. Roughly speaking, this is the condition: there must be a mapping $f: T \to I$, where I

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is a set with a partial order, such that f satisfies certain "reasonable" conditions. See Theorem 1.2 for the precise statement. In the first section we also give a more general criterion which ensures that (for T a noetherian topological space) each stratum will be a constructible set (Theorem 1.3).

In Section 2 we give an application to the case of families of smooth surfaces equipped with a coherent sheaf of ideals with finite support. Thus, essentially on each surface we have a finite number of closed points $P_1, \dots, P_n$ and an ideal in the local ring at each P_i, primary to the maximal ideal. The "complexity" of such a configuration is measured by a finite *forest* (a disjoint union of *trees*), where each vertex has attached a certain *weight*. This is obtained by "simplifying" the ideal by means of quadratic transformations and by taking proper transforms of the ideals, the vertices correspond to the infinitely near points that appear, the weights to the orders of the proper transforms of the ideals (alternatively, we could also take into account *proximities*). This will be called the *desingularization forest*. Two points t and t' of the parameter space will be in the same stratum if and only if the desingularization forests of the corresponding geometric fibers are isomorphic. Using the criterion mentioned above, we can prove that the resulting strata are locally closed sets. Following [R], these may be called *equisingular strata*. Here our treatment relies on the behavior of the multiplicity in a family of zero-dimensional ideals, as presented in [L1]. The theory of equisingular families of planar ideals (in the local complex analytic context) was initiated and developed by F. Pham and J. J. Risler in the early 1970s (cf. [R] and the works by Pham mentioned there). The authors of the present paper have discussed these matters in the general algebro-geometric setting in [NV].

In Section 3 we specialize the discussion of Section 2 to the important case of certain universal families provided by the theory of Hilbert schemes. We obtain in this way families of ideals (of the type discussed above), universal with respect to families whose geometric fibers have a constant desingularization forest (Theorem 3.1). These families have specially nice properties when the ideals that appear are *integrally closed*, or *complete*; these are also studied in Section 3.

In an Appendix (Section 4) we gather basic results about families of ideals on surfaces in particular equivalences of the condition "the geometric fibers have a constant desingularization forest." These results are essential for certain proofs in the preceding sections. They are primarily an adaptation to the abstract algebro-geometric set-up of results of Risler (cf. [R], where he works in the local complex analytic case). To our knowledge, they do not appear in the literature in the present form; they might be of some independent interest.

We hope that the techniques of this paper will find other applications, for instance to show that natural stratifications of parameter spaces of certain families of higher dimensional singularities involve locally closed strata. For example, we have in mind stratifications such that the induced family on each stratum admits, in a suitable sense, a simultaneous desingularization (the results here correspond, in a sense, to the zero-dimensional case). We plan to discuss these matters in future works.

In this paper we use the standard language and notation of the theory of schemes as explained, for instance, in Hartshorne's book [H]. There are a few exceptions, for instance sometimes a coherent sheaf of ideals I on a scheme X will be called an *X-ideal*. If I is an X-ideal, the subscheme of X defined by I will be denoted by $V_s(I)$ while the corresponding reduced one by $V(I)$.

1. TOPOLOGICAL RESULTS

Let T be a Zariski space (i.e., a noetherian topological space such that each irreducible closed subspace T' of T has a unique generic point, cf. [H, p. 93]) and $(I, \leq)$ a partially ordered set. If $a \leq b$ but $a \neq b$, we shall write $a < b$. If $f: T \to I$ is a mapping and $Y \subseteq T$ is a subspace, the restriction of f to Y will be denoted by f_Y.

We shall introduce next a class of functions from T to I which, as will be seen in Theorem (1.2), have the property that the fibers are locally closed sets of T; for this reason they will be called *LC-functions*. Precisely, an *I-valued LC-function on T* is a mapping $i: T \to I$ such that:

(L_1) If x, y are points of T, $y \in \bar{x}$ (the closure of x in T), then $i(y) \geq i(x)$.

(L_2) For every irreducible closed subspace T' of T, one of the fibers of the restriction $i_{T'}$ contains a non-empty (hence dense) open set of T'.

Then we have, more precisely,

PROPOSITION (1.1). *Let i be an I-valued LC-function on the irreducible Zariski space T, $x \in T$ its generic point. Then, the set $\{t \in T: i(t) = i(x)\}$ is open in T.*

Proof. By condition (L_1), every $t \in T$ satisfies $i(t) \geq i(x)$. Consider $C = \{t \in T: i(t) > i(x)\}$. We shall see that this set is closed in T. Let D be its closure. Write $D = D_1 \cup \cdots \cup D_n$ (union of irreducible components) and let x_i be the genetic point of D_i. We claim that, for all j, $x_j \in C$. In fact, by (L_2), there is a dense open set U_j of D_j such that for all $z \in U_j$ we have $i(z) = i(x_j)$. But clearly $C \cap D_j$ is dense in D_j, hence $C \cap U_j \neq \emptyset$.

Thus $i(x_j) > i(x)$ and so $x_j \in C$, as claimed. But if w is in D_j, then by L_1 we have $i(w) \geq i(x_j) > i(x)$. So, $w \in C$. Thus, $D_j \subseteq C$, for all j, hence $D = C$ and C is closed. ∎

THEOREM (1.2). *Let i be an I-valued LC-function on the Zariski space T. Then, each fiber of i is a locally closed set of T.*

Proof. Consider a non-empty equivalence class, say $C = \{t \in T: i(t) = \gamma\}$, for a certain $\gamma \in I$. Let D be the closure of C, $D_1, \dots, D_n$ its irreducible components. Let $S_j = \{t \in D_j: i(t) \neq \gamma\} = D_j - C$. We shall see that each S_j is closed in D_j (and hence in D). Since $S_1 \cup \cdots \cup S_n = \{t \in D; i(t) \neq \gamma\} = D - C$, this will prove the theorem. Let x_j be the genetic point of D_j. By (L_2), there is a dense open set V in D_j such that all $t \in V$ satisfies $i(t) = i(x_j)$. Since $C \cap D_j$ is dense in D_j, $C \cap V \neq \emptyset$ and hence $i(x_j) = \gamma$. But by Proposition (1.1) of this section, $\{t \in D_j: i(t) = i(x_j) = \gamma\}$ is open in D_j, thus S_j is closed in D_j, as claimed. ∎

Under weaker hypotheses we still can get some results. For instance, we have the following theorem (recall that a subset of a noetherian topological space is constructible if it can be expressed as a finite union of locally closed sets, cf. [H, p. 94]).

THEOREM (1.3). *Let T be a noetherian topological space, S a set, $i: T \to S$ a function, and assume that for each closed irreducible subspace Y of T a fiber of the restriction $i_Y: Y \to S$ contains a non-empty open set of Y. Then, every fiber of i is a constructible set.*

Proof. By contradiction, were the conclusion false, then the set $L = \{Y \subseteq T: Y$ is closed and some fiber of i_Y is not constructible$\}$ is non-empty (because at least T is in this set). By the noetherian hypothesis, there is a minimal Y in L. There are two possibilities; we'll see that each one leads to a contradiction.

(i) *Y is irreducible*. Then, by assumption, a fiber of i_Y contains a dense open set, hence there is an open set U of Y such that i_Y is constant on U. Let $Y' = Y - U$. Then, Y' is properly contained in Y and by minimality of Y, all fibers of i'_Y are constructible subsets of Y' (and hence of Y). Thus, $Y' = \bigcup_{j=1}^{n} Y'_j$, a disjoint union where the restriction of i to each T_j is constant. Then, the expression $Y = U \cup \bigcup_{j=1}^{n} Y'_j$ shows that all fibers of i_Y are constructible, a contradiction.

(ii) *Y is reducible*. Then $Y = V \cup W$, a union of two properly contained closed subsets. By minimality of Y, all the fibers of both i_Y and i_W are constructible sets. Thus, $V = \bigcup_{j=1}^{n} V_j$ (resp. $W = \bigcup_{q=1}^{m} W_q$), where i_{V_j} (resp. i_{W_q}) is constant, for all j (resp. all q). Similarly, $F = V \cap W$ is a closed subset of Y, $F \neq Y$, hence $F = \bigcup_{p=1}^{r} F_p$, where each F_p is locally closed and i_{F_p} is constant, and all these are disjoint unions. Let $V_j' = V_j \cap (V - W)$, $W_q' = W_q \cap (W - V)$, for all j, q. Then the expression

$$Y = \bigcup_{j=1}^{n} V_j' \cup \bigcup_{q=1}^{m} W_q' \bigcup_{p=1}^{r} F_p$$

shows that all the fibers of i_Y are constructible, a contradiction. ∎

2. A GEOMETRIC APPLICATION

Consider a smooth morphism $\pi : Z \to T$, where T is a noetherian scheme and all the fibers are purely two-dimensional, and a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$, such that the induced morphism $\mathcal{V}(\mathcal{I}) \to T$ is finite. The pair $(\pi, \mathcal{I})$ (or simply $(Z, \mathcal{I})$, if T and π are clear) will be called *a family of zero-dimensional ideals on surfaces, parameterized by* T. Henceforth, this will be referred to simply as a *family of ideals*.

If T is a scheme and t a point of T, a geometric point of T at t is a morphism spec$(k) \to T$ landing at t, where k is an algebraic closure of the residue field $k(t)$ of T at t. Given $Z \to T$ as above, the fiber at a geometric point $\bar{t}$ at $t \in T$ will be called "the geometric fiber over t," denoted by $Z_{\bar{t}}$. Although there are different algebraic closures of $k(t)$, all the corresponding fibers will be isomorphic, so that our terminology is justified. If $(Z, \mathcal{I})$ is a family of ideals parametrized by T, let $\mathcal{I}_{\bar{t}}$ be the sheaf of ideals induced in $\mathcal{O}_{Z_{\bar{t}}}$ (also called the geometric fiber of $\mathcal{I}$ at t). Similarly, Z_t will denote the fiber $\pi^{-1}(t)$ and $\mathcal{I}_t$ will be the sheaf of ideals induced by $\mathcal{I}$ on Z_t.

Remarks (2.1). (a) If F is a smooth algebraic surface over a field k and $\mathcal{J}$ is a sheaf of ideals on F having finite support $\mathcal{V}(\mathcal{J})$, it is well known that we may "resolve" $\mathcal{J}$ by means of quadratic transformations. More precisely, if we blow-up F with center $\mathcal{V}(\mathcal{J})$, take the proper transform $\mathcal{J}_1$ of $\mathcal{J}$, and repeat the process, eventually we reach a situation where the proper transform of the ideal has empty support. For the definition of *proper transform* see, for instance, [ZS, p. 367; L3, Sect. 1; L2, (1.5)]. Thus, we may attach to $\mathcal{J}$ a finite weighted forest (disjoint union of weighted trees); the vertices are the points on the support of $\mathcal{J}$ or any of its transforms, two vertices being joined by an edge if one is obtained from the other by a single quadratic transformation, the weight is the order of the proper transform of $\mathcal{J}$ at that point. (The notion of *order* is recalled in (4.2)(b).)

We say that a vertex Q of such a forest is of *level* n if one has to traverse n edges to descend from Q to the root of the corresponding tree (thus, a root has level zero). The maximum level of a vertex of the forest is called the *height* of the forest; it will be denoted by $N(\alpha)$. In a similar way, one may consider the resolution forest of $\mathcal{J}$ but where now we associate to each vertices two weights, one as before (corresponding to the order of the proper transform) and the other to the *proximity index* of the point, this will be the *bi-weighted resolution forest* of $\mathcal{J}$. For the notion of *proximate point* see [C, L2]. The proximity index of a point Q of level 0 in the forest of $\mathcal{J}$ is 0, and if the level of Q is $n > 0$ then it is the largest integer r such that Q is proximate to a point in the forest of level $n - r - 1$.

(b) If the field k is not algebraically closed, the different points in the support of $\mathcal{I}$, or infinitely near to one of these points, in general will not be rational over k; the corresponding degree extensions should be recorded. However, for us the most interesting case will be that where the base field is algebraically closed (our surfaces will be the geometric fibers of a family $(\pi, \mathcal{I})$ as above). Anyway, note that if k is arbitrary, but all the points corresponding to vertices of the forest of $\mathcal{I}$ are k-rational, then the weighted (or bi-weighted) forests of $\mathcal{I}$ and of $\mathcal{I}'$, the sheaf of ideals induced by $\mathcal{I}$ on $\mathcal{F}' = \mathrm{spec}(\bar{k}) \times_{\mathrm{spec}(k)} F$ (where $\bar{k}$ is an algebraic closure of k), are canonically isomorphic.

(c) A weighted forest which is (isomorphic to) one of this form (for a suitable choice of a surface F smooth over an algebraically close field k and a sheaf of ideals $\mathcal{J}$) will be called a *geometric weighted forest*, and similarly for bi-weighted forests.

(d) Also note that if k is algebraically closed and K is any algebraically closed extension of k, the forest of $\mathcal{I}$ and that of the induced sheaf of ideals after base extension are canonically isomorphic.

(e) Finally, given a geometric forest, we attach to each vertex P_i a positive integer m_i, the *multiplicity at* P_i, as

$$m_i = \sum_j \nu_j^2$$

where ν_j is the weight at Q_j, and the sum runs over $\{j : Q_j$ lies over P_i in the forest (P_i included)$\}$. According to (4.2)(b), if our geometric forest is that associated to a finitely supported sheaf of ideals $\mathcal{I}$ over the surface F, then m_i is the multiplicity of the stalk of the appropriate proper transform of $\mathcal{I}$ at the point P_i.

Let B (resp. B') denote the set of isomorphism classes of geometric weighted (resp. bi-weighted) forests.

For each family of ideals $(\pi: Z \to T, \mathcal{I})$ as above we introduce a function b from $\mathrm{sp}(T)$ to B, where $\mathrm{sp}(T)$ denotes the underlying topological space of the scheme T, as in [H, p. 93], defined as

$$b(t) = \text{class of the geometric forest of } \mathcal{I}_{\bar{t}} \subset \mathcal{O}_{Z_{\bar{t}}}.$$

This mapping b will be called the *RF-function of the family* $(\pi, \mathcal{I})$ (the choice of the name is due to the use of the resolution forests of the fibers of $\mathcal{I}$ in its definition).

Now assume that T *is an excellent noetherian scheme*. We shall see that it is possible to introduce a partial order $\preccurlyeq$ in B, so that b becomes a B-valued LC-function on T.

First, let us introduce a relation $\ll$ ("less or equal") on B, as follows: if α, β are in B, we put $\alpha \ll \beta$ if there is a family of ideals, parametrized by $V = \mathrm{spec}(R)$, with R an excellent discrete valuation ring, such that $b(x) = \alpha$ and $b(y) = \beta$ (where x and y are the generic and special points of V, respectively).

We would like to see that $\ll$ extends to a partial order in B. For α, β in B, put $\alpha \preccurlyeq \beta$ if there is a chain (in B) $\alpha = \alpha_1 \ll \alpha_2 \ll \cdots \ll \alpha_n = \beta$, for some n. Then,

PROPOSITION (2.2). *The relation $\preccurlyeq$ is a partial order in B.*

Proof. It suffices to show that given a chain $\alpha = \alpha_1 \ll \alpha_2 \ll \cdots \ll \alpha_n = \beta$ (for some positive integer n), if for some i we have $\alpha_i \neq \alpha_{i+1}$, then $\alpha \neq \beta$. To check that this is the case, we introduce an auxiliary function from B to $\mathbb{N}$, the set of all sequences of integers, ordered lexicographically, as follows: if α is a geometric forest,

$$j(\alpha) = (m_0, c_0, \nu_0, m_1, c_1, \nu_1, \ldots, m_h, c_h, \nu_h),$$

where h is the height of α, $-c_i$ = number of vertices of $b(\alpha)$ of level i, m_i = sum of the multiplicities at vertices of $j(\alpha)$ of level i (cf. (2.1)(e)) and ν_i = sum of the weights at vertices of $j(\alpha)$ of level i. Then the proposition immediately follows from:

LEMMA (2.3). *Let $\alpha \ll \beta$. Then* (1) $j(\alpha) \leq j(\beta)$ *and* (2) *if moreover* $j(\alpha) = j(\beta)$, *then* $\alpha = \beta$.

Proof of Lemma (2.3). Let the relation $\alpha \ll \beta$ be "realized" by the family $(W \xrightarrow{\pi} V, \mathcal{I})$ with V the spectrum of a discrete valuation ring R. Let τ be the generic point of V, 0 the special one, and $K = k(\tau)$. We may assume without loss of generality that each point of $\mathbf{V}(\mathcal{I}_\tau)$ is K-rational (in fact, these points are defined over some finite extension K' of K; let

R' be the normalization of R in K' (again a noetherian ring, by the excellence of R), then replace R, if necessary, by a suitably localization of R' at a height-one prime ideal). Let $j(\alpha) = (m_0, c_0, \nu_0, m_1, c_1 \dots)$, $j(\beta) = (m'_0, c'_0, \nu'_0, m'_1, c'_1 \dots)$. Then, by (4.1), $m'_0 \geq m_0$, and there are two possibilities:

(I) $m_0 < m'_0$,

(II) $m_0 = m'_0$.

If (I) holds, then $j(\alpha) < j(\beta)$ (because of the lexicographical order on $\mathcal{N}$), thus (1) is true. If (II) holds then we claim that $-c_0 = \#\{\text{vertices of level zero of } \alpha\} \geq \#\{\text{vertices of level zero of } \beta\} = -c'_0$. We prove this in the following parts (A) and (B).

(A) First we check that the induced morphism $\pi' \colon V(\mathcal{I}) \to V$ is flat. Indeed, since both schemes are one-dimensional and V is regular, this is equivalent to saying that $V(\mathcal{I})$ has no isolated points (the only possible ones will belong to the closed fiber $W_0 = \pi^{-1}(0)$). Let $Q_1, \dots, Q_r$ be the non-isolated points of $V(\mathcal{I})$ (namely, the points of $V(\mathcal{I}) \cap W_0$ which are not isolated points of $V(\mathcal{I})$), $Q_{r+1}, \dots, Q_s$ the isolated ones. Thus, $V(\mathcal{I}_0) = \{Q_1, \dots, Q_r, Q_{r+1}, \dots, Q_s\} \subset W_0$. Assume, by contradiction, $s > r$. For $i = 1, \dots, r'$, let P_i be a point of $V(\mathcal{I}_\tau)$ in W_τ (the generic fiber) such that its closure S_i contains some point Q_j $(1 \leq j \leq r)$. By our rationality assumption, the induced projection $S_i \to V$ is birational. Since V is the spectrum of a discrete valuation ring it follows that $S_i \to V$ is an isomorphism, in particular $S_i \cap W_0$ consists of a unique point $Q_{j(i)}$, $1 \leq j(i) \leq r$. Let $S = S_1 \cup \cdots \cup S_{r'}$, each S_i with generic point P_i (as before) and set $Z = \{Q_{r+1}, \dots, Q_s\}$ so that $V(\mathcal{I}) = S \cup Z$, $S \cap Z = \emptyset$, and Z consists of isolated points of $V(\mathcal{I})$. In particular, $Z_\tau = \emptyset$. Let $\mathcal{J}$ be the sheaf of ideals defined by

$$\begin{aligned} \mathcal{J}_x &= \mathcal{I}_x & \forall x \in W - Z \\ \mathcal{I}_x &= \mathcal{O}_{W,x} & \forall x \in Z. \end{aligned}$$

Then $\mathcal{I} \subseteq \mathcal{J}$, $\mathcal{J}$ defines a family of ideals, and $\mathcal{J}_\tau = \mathcal{I}_\tau$ (they coincide at the generic point). Hence it follows that both S and $V(\mathcal{I})$ define the same set of closed points of W_τ. Hence, $m_0(\mathcal{I}) = m_0(\mathcal{I})$. By (4.1) we have $m_0 \leq m'_0(\mathcal{J}) := \sum_{i=1}^{r} e(\mathcal{J}_{0,Q_i})$ (note that all the points involved are rational). Since $\mathcal{J}_{0,q_i} = \mathcal{I}_{0,Q_i}$, $i = 1, \dots, r$, it follows that

$$m_0(\mathcal{I}) = m_0(\mathcal{J}) \leq \sum_{i=1}^{r} e(\mathcal{I}_{0,Q_i}) < \sum_{i=1}^{s} e(\mathcal{I}_{0,Q_i}) = m'_0(\mathcal{I}),$$

contradicting (II). Thus, there are no isolated points.

(B) It follows from (A) that if $m_0 = m_0'$, the closed fiber $W_0 \cap \mathcal{V}(\mathcal{I})$ has no isolated points, so that $\mathcal{V}(\mathcal{I}) = S_1 \cup \cdots \cup S_{r'}$ and $\{Q_1, \dots, Q_{r+1}, \dots, Q_s\}$. Since $Q_j = Q_{j(i)} = S_i \cap W_0$, it also follows that $r \le r'$; so that $c_0 = -r' \le -r = c_0'$.

Now we analyze the behavior of the third coordinate of $j(\alpha)$ and $j(\beta)$ under the assumption that their first and second coordinates are respectively equal. Since V is regular and $\pi: W \to V$ is smooth, it follows that the order of $\mathcal{I}$ at points of W defines an upper-semicontinuous function $\mathrm{sp}(W) \to \mathbf{Z}$. This, together with the assumption $c_0' = c_0$, implies that (letting as above $\{P_1, \dots, P_r\}$ be the generic fiber and Q_i the specialization of P_i, for all i)

$$\nu(\mathcal{I}_{\tau, P_i}) \le \nu(\mathcal{I}_{0, Q_i}). \tag{2.3.1}$$

So, clearly $\nu_0 \le \nu_0'$, and equality holds if and only if it holds in (2.3.1) for each index $i = 1, \dots, r$. If $\nu_0 = \nu_0'$ let W_1 be the blowing-up of W along $\mathcal{V}(\mathcal{I})$, $\mathcal{I}_1$ the proper transform of $\mathcal{I}$. As in the Appendix, (4.4), we see that these satisfy the original hypotheses, and that if α' and β' are the geometric forests of the generic and special fibers, respectively, their j's are $j(\alpha)$ and $j(\beta)$, with the first three terms deleted. So we may repeat, or use induction on the height, to get (1). The proof of (2) is similar: in this case, definitely $m_0 = m_0'$, $\nu_0 = \nu_0'$, and $c_0 = c_0'$, so when we blow-up $\mathcal{V}(\mathcal{I})$ we obtain, in the above notation, $j(\alpha') = j(\beta')$, and we may use the induction hypothesis. ∎

In view of Proposition (2.2) it makes sense to state:

THEOREM (2.4). *Let $(Z, \mathcal{I})$ be a family of ideals parametrized by an excellent noetherian scheme T. Then, the corresponding RF-function $b: T \to B$ is a B-valued LC-function on T.*

Proof. Let us check property (L_1). Consider two points y, w in $T \in \mathcal{C}$, with $y \in \overline{w} = W$. Let $Y = \overline{y}$. Blow-up the integral scheme W along Y, and then normalize, to get a morphism $\pi: W' \to W$. By the excellence of T, W' is a noetherian scheme. The resulting exceptional divisor E has at least an irreducible component D (a prime Weil divisor of the normal scheme W') such that $\pi(D) = Y$. Let y' (resp. z) be the generic point of D (resp. of Z). Then, $\pi(y') = y$ and $\pi(z) = w$. Let $\mathcal{I}'$ be the sheaf of ideals on W' induced by $\mathcal{I}$ and b' the RF-function of the family $(W', \mathcal{I}')$. One easily checks that $b(w) = b'(z)$ and $b(y) = b'(y')$. But consider $R = \mathcal{O}_{Z, y'}$, which is a discrete valuation ring. Let y_1 and z_1 be the special and generic points of $\mathrm{Spec}(R)$, respectively. One easily checks, taking into account the comments at the end of Remark (2.1)(b), that $b_R(y_1) = b(y')$ and

$b_R(z_1) = b(z)$, respectively (where b_R is the RF-function corresponding to the family induced by pulling back over Spec R). By the definition of "$\ll$", $b_R(y_1) \succcurlyeq b_R(z_1)$. By the equalities just listed, $b(y) \succcurlyeq b(w)$, as we wanted to prove. ∎

Property (L_2) is a consequence of the following lemma, whose proof will be presented in the Appendix (Corollary (4.8)).

LEMMA (2.5). *Let T, Z, $\mathcal{I}$ be as above, with T integral. Then there is a dense open set U of T such that for all $t \in U$, $b(t) = \gamma$, where x is the generic point of T and $b(t) = \gamma$.*

With minor modifications, we may repeat what was done above in the case where the function we use is not b, but rather

$$b'(t) = \text{class of the bi-weighted resolution forest of } \mathcal{I}_{\bar{t}} \subset \mathcal{O}_{Z_{\bar{t}}},$$

where the bi-weights, as usual, are the order of the proper transform and the proximity index. (See Corollary (4.10) for the appropriate version of Lemma (2.5).)

Summarizing, we have:

THEOREM (2.6). *If $(Z, \mathcal{I})$ is a family of ideals parametrized by an excellent noetherian scheme T, then there is a partition $\{T_i'\}$, $i \in A'$ (resp. $\{T_j''\}$, $j \in A''$) into locally closed subsets of T, such that for points t and t' in T we have $b(t) = b(t')$ (resp. $b'(t) = b'(t')$) if and only if t and t' are in the same stratum T_i (resp. T_j').*

3. FURTHER APPLICATIONS

In the sequel, we work primarily with the RF-function b and the ordered set B introduced in Section 2, but a similar discussion applies if we use b' and the ordered set B'. The expression "family of ideals" will always mean "family of zero-dimensional ideals," in the sense described at the beginning of Section 2. As before, if $\mathcal{I}$ is a sheaf of ideals on a scheme X, $\mathcal{V}_s(\mathcal{I})$ will denote the corresponding closed subscheme of X and $\mathcal{V}(\mathcal{I}) := \mathcal{V}_s(\mathcal{I})_{\text{red}}$.

Fix a scheme X, projective and smooth over spec($\mathbf{Z}$), of relative dimension two, n a positive integer and an element α of B. If T is a scheme, let $X_T = X \times_{\text{spec}\,\mathbf{Z}} T$. Let $\mathcal{F}(X, \alpha, n) := \mathcal{F}$ be the class of all families

$(\pi: X_T \to T, \mathcal{I})$ of ideals satisfying the following conditions:

(0) T is a reduced, excellent noetherian scheme.

(i) For all $t \in T$, $b(t) = \alpha$ (hence, the family satisfies condition (b) of (4.6)).

(ii) The induced morphism $\mathcal{V}_s(\mathcal{I}) \to T$ is flat.

(iii) For every geometric point $\bar{t}$ of T, $\dim_{k(\bar{t})}(\mathcal{O}_{X_{\bar{t}}}/\mathcal{I}_{\bar{t}}) := \sum_{j=1}^{r} \dim_{k(\bar{t})}(\mathcal{O}_{X_{\bar{t}, z_j}}/\mathcal{I}_{\bar{t}, z_j}) = n$ (where $\{z_1, \ldots, z_r\}$ is the support of $\mathcal{I}_{\bar{t}}$, a finite set of points).

Then we have:

THEOREM (3.1). *There is a universal object for families in $\mathcal{F}$. That is, there is a family $(p: X_{H_{\alpha n}} \to H_{\alpha n}, \mathcal{I}_{\alpha n})$ such that for each family $(\pi, \mathcal{I})$ in $\mathcal{F}$, there is a unique morphism $T \to H_{\alpha n}$ such that $(\pi, \mathcal{I})$ is the pull-back of $(p, H_{\alpha n})$.*

Proof. Consider the Hilbert scheme H, parametrizing subschemes of X whose fibers have Hilbert polynomial n. Let $Z \subset X \times_{\operatorname{spec} \mathbf{Z}} H = X_H$ be the universal family; note that Z corresponds to an X_H-ideal $\mathcal{I}$. Now, the pair $(Z, \mathcal{I})$ defines a family of ideals (where moreover the projection $\mathcal{V}_s(\mathcal{I}) \to H$ is flat). We apply Theorem (2.6) to this family $(Z, \mathcal{I})$, where we consider b as our LC-function. This leads to a partition of H into a disjoint union of locally closed sets C_j, $j = 1, \ldots, m$, where two geometric fibers $\mathcal{I}_{\bar{t}}$ and $\mathcal{I}_{\bar{t}'}$ have the same weighted tree if and only if t and t' belong to the same stratum C_j. Thus, b induces a natural injective mapping $\{C_j, j = 1, \ldots, m\} \to B$; let us denote by $H_{\alpha n}$ the stratum whose image is $\alpha \in B$. Now, it is easy to see that given a family $(\pi, \mathcal{I})$ in $\mathcal{F}$ (or, equivalently the corresponding flat family of subschemes $(\pi, \mathcal{V}_s(\mathcal{I}))$, the natural morphism $T \to H$ obtained by universality of the Hilbert scheme factors through $H_{\alpha n}$, and vice versa, proving the theorem. ∎

Remark. In other words, Theorem (3.1) says that the functor, from the category of excellent noetherian reduced schemes (sets) (with the usual morphisms) associating to a scheme T the set of families of ideals $(\pi: X_T \to T, \mathcal{I})$ satisfying conditions (0) to (iii) above, is representable.

The situation becomes nicer when we deal with complete, or integrally closed, ideals. Let us say that a family of ideals $(W \to U, \mathcal{J})$ is *complete* if all the induced ideals $\mathcal{J}_{\bar{u}}$ (over geometric fibers) are integrally closed (i.e., $(\mathcal{J}_{\bar{u}})_z$ is a complete ideal in the local ring $\mathcal{O}_{W_{\bar{u}}, z}$, for all z in the geometric fiber $W_{\bar{u}}$. Then we have:

PROPOSITION (3.2). *Let $(Z \to T, \mathcal{I})$ be a complete family of ideals, with T reduced, such that the corresponding mapping $b: T \to B$ is constant, say $= \alpha$ (hence, it satisfies condition* (b) *of* (4.6)). *Then* (a) *the induced mor-*

phism $V_s(I) \to T$ *is flat*, (b) *the constant number* $\dim_{k(\bar{t})}(\mathcal{O}_{X_{\bar{t}}}/I_{\bar{t}})$, *where* $\bar{t}$ *is any geometric point of* T, *is determined by* α.

Proof. Since T is reduced and $V_s(I) \to T$ is finite, the flatness of this morphism is equivalent to the constancy of the number $\dim_{k(\bar{t})}(\mathcal{O}_{X_{\bar{t}}}/I_{\bar{t}})$, for $\bar{t}$ any geometric point of T. But recall the H−D (Hoskin−Deligne) formula: if J is an M-primary complete ideal in a regular two dimensional local ring (A, M, k), then $l(A/J) = \Sigma_P[k(P): k]\binom{\nu_P + 1}{2}$, over points P infinitely near to M, where ν_P denotes order of the proper transform and l denotes "length" (as an A-module) (see [L3, Theorem (3.1), JV]). Because of condition (b) (see (4.6), the numbers ν "are constant," whence both parts of the statement follow. ∎

Remark (3.3). The following is a useful observation, which immediately follows from the formula of Hoskin and Deligne (and which we state in the only case interesting to us). If (A, M) is a regular two-dimensional local ring, where $A/M = k$ is an algebraically closed field, I is an ideal of A, then I is integrally closed if and only if $\dim_k(A/J) = \Sigma_p\binom{\nu_P + 1}{2}$ (for the implication $\Leftarrow$, note that $l(A/J) = \dim_k(A/J)$; consider $I \subseteq \bar{I}$ and use the H−D formula to see that the equality must hold).

Under suitable assumptions, a family cannot be "partially complete," as the following result shows.

PROPOSITION (3.4). *Let* $(Z \to T, I)$ *be a family of ideals, with* T *reduced, satisfying conditions* (b) *of* (4.6) *and* (iii) (*see the paragraph preceding Theorem* (3.1)). *Assume that for some geometric point* $\bar{u}$ *of* T *the ideal* $I_{\bar{u}}$ *is complete. Then,* $I_{\bar{t}}$ *is complete for every geometric point* $\bar{t}$ *of* T (*i.e., the family is complete*).

Proof. By a suitable finite base change, we may assume that our family is resolvable by sections (cf. Theorem (4.8)). Choose the indices in such a way that if t and u are points of T, $\nu_{t,i}$, $\nu_{u,i}$ are the weights at points lying over t and u, respectively, in the image of the same section (thus, by (b), they are equal); similarly for geometric points. Now, let $\bar{t}$ be a geometric point of T, let $\overline{I_{\bar{t}}}$ be the integral closure of $I_{\bar{t}}$. Then $\dim_k(\mathcal{O}_{X_{\bar{t}}}/I_{\bar{t}}) \geq \dim_k(\mathcal{O}_{X_{\bar{t}}}/\overline{I_{\bar{t}}}) = \Sigma_i\binom{\nu_{\bar{t},i} + 1}{2} = \Sigma_i\binom{\nu_{\bar{u},i} + 1}{2} = \dim_k(\mathcal{O}_{X_{\bar{u}}}/I_{\bar{u}})$ (the last equality because of the completeness assumption and the H−D formula). Since the first and last terms are equal (by (iii)), all these are equalities; hence (by (3.3)) $\overline{I_{\bar{t}}} = I_{\bar{t}}$. ∎

Next we shall discuss further properties of complete families.

PROPOSITION (3.5). *Let* F *be a smooth algebraic surface over a field* k, L *the algebraic closure of* k, $F' = F \times_k L$, I *an ideal on* F, *and* $J' = J\mathcal{O}_{F'}$.

Then, if $\mathcal{J}'$ *is complete,* $\mathcal{J}$ *is also complete. Conversely, if* $\mathcal{J}$ *is complete and the field* k *is perfect, then* $\mathcal{J}'$ *is complete.*

Proof. For the first part, taking into account the fact that a field extension is always faithfully flat, the result rapidly follows from the following algebraic considerations. If $A \subset B$ is an inclusion of rings and $J \subset B$ is integrally closed, then $J \cap A$ is an integrally closed ideal of A (an elementary fact); moveover if B is a faithfully flat A-algebra and $I \subset A$ is an ideal, then $IB \cap A = I$ [M, p. 49].

Concerning the converse, let $\{P_1, \ldots, P_r\}$ be all the points, on F or infinitely near to it, which are in the support of $\mathcal{J}$. Let k' be a finite extension of k (contained in L) such that $P_1, \ldots, P_r$ are all defined over k'. Then, the induced map $F \to F_1 := F \times_k k'$ is etale, and hence $\mathcal{I}_1 := \mathcal{J}\mathcal{O}_{F_1}$ is complete (cf. [L4, Remark (e), p. 660] and to check that the hypotheses of this remark are valid in our situation, use [G, Proposition 6.14.4]). Hence, since there are no non-trivial field extensions, by the Hoskin–Deligne formula $\dim_{k'}(\mathcal{O}_{F_1}/\mathcal{J}_1) = \sum_{j=1}^{r} \binom{\nu_{P_j}+1}{2}$ (where we identify the P_j's with the corresponding points of F_1). Clearly, the latter number is also equal to $\dim_L(\mathcal{O}_{F'}/\mathcal{J}')$; now the result follows from Remark (3.3). ∎

In Proposition (3.5) (converse), the assumption on the residue field being perfect is necessary, otherwise these are counterexamples.

The following corollary is immediate.

COROLLARY (3.6). *Let* $(Z \to T, \mathcal{I})$ *be a family of ideals, with* T *reduced. If this is a complete family, then for each point* $t \in T$ *the* $\mathcal{O}_{X_t}$*-ideal* $\mathcal{I}_t$ *is complete. Conversely, if we assume all residue fields of* T *perfect, if* $\mathcal{I}_t$ *is complete for all* t *in* T*, then the given family of ideals is complete.*

Recall that given a sheaf of ideals $\mathcal{J}$ on a scheme W it is possible to construct the *integral closure* of $\mathcal{J}$. This is a sheaf $\bar{\mathcal{J}}$ such that $\bar{\mathcal{J}}_w = \overline{\mathcal{J}_w}$ (equality of stalks), for each $w \in W$. Locally, over an affine open spec(A), if $\mathcal{J}$ corresponds to the ideal $J \subset A$, then $\bar{\mathcal{J}}$ corresponds to the integral closure of $\mathcal{J}$. The sheaf $\mathcal{J}$ is said to be *integrally closed* if it coincides with its integral closure.

Then we have the following result.

THEOREM (3.7). *Let* $(Z \to T, \mathcal{I})$ *be a family of ideals, with* T *reduced. Then, if this is a complete family, the sheaf* $\mathcal{I}$ *is integrally closed. Conversely, if we assume that all the residue fields of* T *are perfect and that* $\mathcal{I}$ *is integrally closed, then the given family of ideals is complete.*

Proof. Assume the given family is complete. Consider the natural exact sequence

$$0 \to \mathcal{I} \to \bar{\mathcal{I}} \to \bar{\mathcal{I}}/\mathcal{I} \to 0.$$

If $t \in T$, we get an induced inclusion $I_{\bar{t}} \subseteq \overline{I}_{\bar{t}}$, and it is easy to see that $\overline{I}_{\bar{t}}$ is integral over $I_{\bar{t}}$. Since $I_{\bar{t}}$ is complete, it must be $I_{\bar{t}} = \overline{I}_{\bar{t}}$. Thus, $\overline{I}/I_{\bar{t}} = 0$. Since $\mathrm{Supp}(\overline{I}/I)$ is finite over T, by Nakayama's Lemma we get $\overline{I}/I = 0$, thus $I = \overline{I}$ as claimed.

Now assume that I is integrally closed (and the residue fields of T are perfect). According to Proposition (3.4), it suffices to prove that there exists a point $t \in T$ such that, on the corresponding geometric fiber, the ideal $I_{\bar{t}}$ is complete. But let u be the generic point of any irreducible component of T. Using the algebraic fact that if I is a complete ideal in a ring A and S is a multiplicative set in A then $IS^{-1}A$ is complete, one readily checks that I_u is complete. Hence, by Proposition (3.5), $I_{\bar{u}}$ will be complete, and the theorem is proved. ∎

Remark (3.8). In [N, Sect. 4], results similar to those of this section are proved in the context of Local Complex Analytic Geometry.

4. APPENDIX: BASIC RESULTS ON FAMILIES

In this appendix, which is logically independent of the rest of this paper, we collect some basic results about families of ideals on surfaces, parametrized by a reduced noetherian scheme, which are needed in different parts of the preceding sections. The main results are inspired by those in Risler's paper [R], where one works with one-parameter families, in the local complex analytic category. The adaptation of Risler's theory to the algebro-geometric setting, in the case where the parameter space is a Dedekind scheme whose residue fields are perfect has been carried out in detail in [NV], where more precise results are obtained.

We begin by recalling some known facts on multiplicities, limiting ourselves to what is essential for our applications.

Let (π, I), $\pi: Z \to T$, a smooth morphism of relative dimension two, be a family of zero-dimensional ideals as in the first paragraph of Section 2. Again, this will be referred to as *a family of ideals,* and we'll use the notation of Section 2.

Let us write, for $t \in T$,

$$\lambda(t, n) = \dim_{k(t)} \left(O_{Z_t} / I_t^{\,n} \right).$$

For n large enough values of this function coincide with those of a polynomial $P(t, n) \in \mathbf{Q}[n]$ of the form

$$P(t, n) = e(t, I) n^2/2 + \cdots$$

for a non-negative integer $e(t, \mathcal{I})$. Then one has:

THEOREM (4.1). *In the setting just described,*

(i) $e(t, \mathcal{I}) = \sum_{q \in S_t}[k(z): k(t)]e_z(\mathcal{I}_t)$, *where* $S_t = \pi^{-1}(t) \cap V(\mathcal{I})$ *and* $e_z(\mathcal{I}_t)$ *is the multiplicity of* $(\mathcal{I}_t)_z$, *an ideal in the local ring* $(\mathcal{O}_{Z_t})_z$.

(ii) *The function* $e: T \to \mathbf{Z}$ *sending* $t \in T$ *to* $e(t, \mathcal{I})$ *is upper-semicontinuous.*

(iii) *e is compatible with base change.*

Proof. Parts (i) and (ii) follow from Propositions 3.1 and 3.3 of [L1, p. 124]. In fact, one can check that conditions (3.3.1) and (3.3.2) of that paper hold in our context (here, the degree $d_P(I)$ of [L1, 3.3] is equal to 2).

Concerning (iii), note that this means that if $\alpha: T' \to T$ is a morphism and $t' \in T'$ maps to $t \in T$, then necessarily

$$e_T(t, \mathcal{I}) = e_{T'}(t', \mathcal{I}'),$$

where $\mathcal{I}' \subset \mathcal{O}_{Z'}$ is defined by base change via α. But the proof of this latter fact is clear from the very formulation of the function $\lambda(t, n)$. ∎

Remark (4.2). (a) Let $\pi: Z \to T$ be smooth of relative dimension two, where now T is an integral scheme [H, p. 92]. A section of π is a morphism $s: T \to Z$ such that $\pi s = \mathrm{id}_T$. Then, the induced projection $s(T) \to T$ is an isomorphism, and the coherent sheaf $\mathcal{P}$ defining the reduced closed subscheme $s(T)$ is a coherent sheaf of ideals $\mathcal{P}$ such that $\mathcal{P}_t \mathcal{O}_{Z_t, s(t)}$ is a regular prime ideal of $\mathcal{O}_{Z_t, s(t)}$, for all $t \in T$. Moreover, under our smoothness assumptions, for all $z \in s(T)$, by tensoring with $k(t)$ the split exact sequence

$$0 \to \mathcal{P}_z \to \mathcal{O}_{Z, z} \to \mathrm{O}_{T, t} \to 0 \qquad (t = \pi(z))$$

it follows that $\mathcal{P}_z = (a, b)\mathcal{O}_{Z, z}$, for suitable elements a, b. Furthermore, $\widehat{\mathcal{O}_{Z, z}} = \widehat{\mathcal{O}_{T, t}}[[x, y]]$ (with x, y analytically independent); in particular a and b form a regular sequence.

(b) Recall that given an M-primary ideal I in a regular two-dimensional local ring (R, M), its order $\nu(I, R)$ is r if I is contained in M^r but not in M^{r+1}. One has the following formula relating the multiplicity $e(I, M)$ and orders: $e(I, M) = \sum [S : R]\nu_S^2$ where S runs over all local rings infinitely near to R, $[S : R]$ denotes the degree of the corresponding residue field extension, and ν_S the order of the proper transform of the ideal I to S (see [JV, Sect. 3] or, in the case where there is no field extension, [R, Sect. 1]; moreover this easily follows from the Hoskin–Deligne formula, using the fact [ZS, p. 385] that, in dimension two, the product of complete ideals is complete).

(c) We shall often be concerned with the following situation: $\pi: Z \to T$ (with T integral) defines a smooth family of surfaces, $s_1, \dots, s_r$ are sections of π, and $s_i(T) \cap s_j(T) = \emptyset$ if $i \neq j$. In this case, the ideal sheaf $\mathcal{A}$ defining $s_1(T) \cup \cdots \cup s_r(T)$ satisfies:

(i) $\mathcal{A} = \mathcal{P}_1 \cdots \mathcal{P}_r$, for suitable ideals $\mathcal{P}_i$, such that the induced morphism $\mathcal{V}_s(\mathcal{P}_i) \to T$ is an isomorphism for all i.

(ii) $\mathcal{O}_Z/\mathcal{A} \approx \mathcal{O}_Z/\mathcal{P}_1 \oplus \cdots \oplus \mathcal{O}_Z/\mathcal{P}_r$.

Here, $\mathcal{P}_i$ is the ideal defining $s_i(T)$. Conversely, given a sheaf of ideals $\mathcal{A}$ satisfying (i) and (ii) (for suitable ideals $\mathcal{P}_i$) then there are sections s_i, $i = 1, \dots, r$ such that $s_i(T) \cap s_j(T) = \emptyset$, and $\mathcal{A}$ defines $s_1(T) \cup \cdots \cup s_r(T)$. Such a Z-ideal $\mathcal{A}$ will be called an *ideal of sections of* π.

PROPOSITION (4.3). *Let $\pi: Z \to T$ define a smooth family of surfaces, $\mathcal{A} \subset \mathcal{O}_Z$ be an ideal of sections, $T' \to T$ a morphism of noetherian schemes, $Z' = Z \times_T T'$, and $\mathcal{A}' = \mathcal{A}\mathcal{O}_{Z'}$. Consider the blowing-up $Z_1 \to Z$ (resp. $Z_1' \to Z'$) of Z along $\mathcal{A}$ (resp. of Z' along A'). Then, there is a natural isomorphism $Z_1' \approx Z_1 \times_Z Z'$.*

Proof. We want to prove that the morphism $Z_1' \to Z_1 \times_Z Z'$ naturally induced by the universal property of the blowing-up [H, p. 164] is an isomorphism. Taking into account the usual description of the blowing-up as a Proj [H, p. 163], it is easy to reduce our problem to the following algebraic statement. Let A and R' be R-algebras of finite type (all noetherian), $A' = A \otimes_R R'$, J an ideal of A such that A/J is isomorphic to R, and J is generated by a regular sequence (a, b) (recall the discussion preceding (4.2)), $\mathcal{R}(J, A) = A \oplus J \oplus J^2 \oplus \cdots$ $\mathcal{R}(J', A') = A' \oplus J' \oplus J'^2 \oplus \cdots$. Then, the natural homomorphism $\mathcal{R}(J, A) \otimes_R R' \to \mathcal{R}(J', A')$ is an isomorphism.

To check this, it suffices to verify that, for all n, the natural homomorphism

$$J^n \otimes_R R' \to J'^n$$

is an isomorphism of R'-modules.

To accomplish this, note that since J is generated by a regular sequence with two elements, $\mathrm{gr}(J, A) = \sum_{n=0}^{\infty} J^n/J^{n+1}$ (direct sum) is isomorphic to a polynomial ring $R[U, V]$ (recall $R = A/J$, see [K, p. 152]). Hence J^n/J^{n+1} is a free R-module, for all n. We have, for all n, an exact sequence of R-modules,

$$0 \to J^n/J^{n+1} \to A/J^{n+1} \to A/J^n \to 0.$$

We can use these and the freeness just established to show, by induction, that A/J^n is a flat R-module, for all n. Then tensoring the exact sequence

$$0 \to J^n \to A \to A/J^n \to 0 \tag{4.3.1}$$

with R' over R we get an exact sequence

$$0 \to J^n \otimes_R R' \to A' \to A'/J'^n \to 0$$

which shows that $J^n \otimes_R R'$ gets identified to J'^n, as wanted. ∎

Remark. We have proved above the flatness of the R-modules A/J^n. If, in addition, we assume that A is flat over R, it follows from (4.3.1) that also $\mathcal{R}(J, A)$ is a flat R-algebra via $R \to A \to \mathcal{R}(J, A)$.

In the next proposition we use the notion of a family of ideals of Section 2 (first paragraph) and the notation in the definition of "ideal of sections" (following Remark (4.2)).

PROPOSITION (4.4). *Let $(Z \xrightarrow{\pi} T, \mathcal{I})$, with T integral, be a family of ideals, $\mathcal{A} = \mathcal{P}_1, \dots, \mathcal{P}_r$ an ideal of sections of π, where the ideal $\mathcal{P}_i$ corresponds to the section $s_i: T \to Z$, $\alpha: Z_1 \to Z$ the blowing-up of Z along $\mathcal{A}$, $\mathcal{I}_1$ the proper transform of $\mathcal{I}$ to Z_1, τ the generic point of T, $s_i(\tau)$ the generic point of $\mathcal{P}_i$. Then* (a) *the composition $\pi_1: Z_1 \to T$ is smooth*, (b) *$(\pi_1, \mathcal{I}_1)$ is a family of ideals if and only if, for all $t \in T$, $i = 1, \dots, r$, $\nu(\mathcal{I}_{t, s_i(t)}, \mathcal{O}_{Z_t, s_i(t)}) = \nu(\mathcal{I}_{\tau, s_i(\tau)}, \mathcal{O}_{Z_\tau, s_i(\tau)})$ (here, ν stands for the order of the indicated ideal in the corresponding ambient local ring, note that these are regular, by* (4.2)(a)). *In this case, $(\mathcal{I}_1)_t$ is the proper transform of $\mathcal{I}_t$.*

Proof. (a) A section $s: T \to Z$ (of $\pi: Z \to T$) is to be thought of as a family of rational points: for each $t \in T$, $s(t)$ is a $k(t)$-rational point of the smooth $k(t)$-surface Z_t. By (4.3), each fiber $(Z_1)_t$ can be identified to Z_t with r rational points blown-up, in particular it is smooth of pure dimension two. It is readily seen (using the Remark after Proposition (4.3)) that π_1 is flat. The result follows [H, Theorem 10.2].

(b) It is easy to reduce it to the case $r = 1$. We write $\mathcal{A} = \mathcal{P}_1 = \mathcal{P}$, $s_1 = s$. Since α is proper, $V(\mathcal{I}_1) \to T$ is proper. Let $\nu = \nu(\mathcal{I}_{\tau, s_i(\tau)}, \mathcal{O}_{Z_\tau, s_i(\tau)})$. Then, $\mathcal{I}_1 = \mathcal{E}^{-\nu} \mathcal{I} \mathcal{O}_{Z_1}$, where $\mathcal{E} = \mathcal{P}\mathcal{O}_{Z_1}$ is the ideal sheaf of the exceptional divisor. As a special case of Proposition (4.3), for $t \in T$ the quadratic transformation of Z_t with center $s(t)$ can be identified to $(Z_1)_t = \pi^{-1}(t)$, the proper transform J of $\mathcal{I}_t$ to $(Z_1)_t$ is $\mathcal{E}_t^{-\nu_t} \mathcal{I}_t(\mathcal{O}_{Z_1})_t$, with $\nu_t = \nu(\mathcal{I}_{t, s_i(t)}, \mathcal{O}_{Z_t, s_i(t)}))$ (by Remark (4.2), $\nu_t \geq \nu$). Hence, $\mathcal{E}^{\nu_t - \nu} J = (\mathcal{I}_1)_t$. Now, since $V(\mathcal{E}_t)$ is a projective line, it becomes clear that $\pi^{-1}(t) \cap V((\mathcal{I}_1)_t)$ will be a finite set if and only if $\nu_t = \nu$. Hence, the induced morphism $V(\mathcal{I}_1) \to T$ will be finite-to-one (that is, by properness, finite) if and only if $\nu_t = \nu$ for all $t \in T$, as claimed. It is clear that in this case $(\mathcal{I}_1)_t$ is the proper transform of $\mathcal{I}_t$. ∎

Now we are in a position to introduce and compare several "equisingularity conditions" that can be imposed on a family of ideals.

DEFINITION (4.5). Let $(\pi: Z \to T, \mathcal{I})$ be a family of ideals (cf. Section 2), where T is integral. Then we shall say that it is *resolvable by blowing-up sections* (or, simply, *resolvable by sections*) if there is a sequence $(Z_i, \alpha_i, \mathcal{I}_i, \mathcal{A}_i)$, $i = 0, 1, \ldots, r$ where (i) $Z_0 = Z$; (ii) $\mathcal{I}_0 = \mathcal{I}$; (iii) $\mathcal{A}_i$ is an ideal of sections on Z_i, and $\mathcal{V}(\mathcal{A}_i) = \mathcal{V}(\mathcal{I}_i)$, for all i; (iv) $\alpha_i: Z_{i+1} \to Z_i$ is the blowing-up of Z_i along $\mathcal{A}_i$, for $= 0, \ldots, r-1$; (v) $\mathcal{I}_{i+1}$ is the proper transform of $\mathcal{I}_i$ to Z_{i+1}, $i = 0, \ldots, r-1$; and (vi) $\mathcal{V}(\mathcal{I}_r) = \varnothing$.

DEFINITION (4.6). Let $(\pi: Z \to T, \mathcal{I})$ be a family of ideals (cf. Section 2). Then we shall say that:

(1) It *satisfies condition* (a) if there is a finite surjective morphism $T' \to T$, with T' a finite disjoint union of integral schemes T_j, $j = 1, \ldots, s$, such that the family induced by $(\pi, \mathcal{I})$ over T_j is resolvable by sections, for all j.

(2) It *satisfies condition* (a$'$) if condition (a) holds and moreover, in the notation above, if for any $j = 1, \ldots, s$, the following is true: consider the sequence of Definition (4.4) (corresponding to the family induced over T_j) and any index $0 \leq i \leq r$. If E is any component of the exceptional divisor of the composition $Z_i \to Z$ and V any irreducible component of $V(\mathcal{A}_i)$, then we have either $E \cap V = \varnothing$ or $V \subseteq E$.

(3) It *satisfies condition* (b) if the function $b: T \to I$ defined by $(\pi, \mathcal{I})$ is constant (cf. Section 2, paragraph following (2.1)(e) for the definition of b).

(4) It *satisfies condition* (b$'$) if the function b' defined by $(\pi, \mathcal{I})$ is constant (cf. Section 2, paragraph following (2.5) for the definition of b').

THEOREM (4.7). *Let* $(\pi, \mathcal{I})$ *be as in Definition* (4.6), *with T connected. Then, this family satisfies condition* (a) *if and only if it satisfies condition* (b).

Proof. (a) *implies* (b). It is easy to reduce it to the case where, in condition (a), $s = 1$, i.e., $T' = T_1$. In other words, without loss of generality we may assume that, in our given family $(Z \xrightarrow{\pi} T, \mathcal{I})$, T is integral, and it is resolvable by blowing-up sections. Then, since (for any $t \in T$) all points in $\mathcal{V}(\mathcal{I}_t)$ or infinitely near to one of these are $k(t)$-rational, we may use the resolution forest of $\mathcal{I}_t$ rather than using the geometric fiber. To show (b), we use induction on the height $N(\gamma)$ (see (2.1)(a)), where γ is the forest of the generic ideal $\mathcal{I}_\tau \subseteq \mathcal{O}_{Z_\tau}$. The case $N(\gamma) = -1$ (i.e., $\mathcal{I}_\tau = \mathcal{O}_{Z_\tau}$) is trivial. Assume next that $N(\gamma) \geq 0$. By condition (a) (and using the notation of Definition (4.3)), $(Z_1, \mathcal{I}_1)$ is a family of ideals. According to Proposition (4.4), this implies $\nu(\mathcal{I}_{t, s_i(t)}, \mathcal{O}_{Z_t, s_i(t)}) = \nu(\mathcal{I}_{\tau, s_i(\tau)}, \mathcal{O}_{Z_\tau, s_i(\tau)})$, for all $t \in T$. Thus we get a bijection of vertices of level zero of the resolution forests at t and τ, preserving weights. Since clearly $(Z_1, \mathcal{I}_1)$ also satisfies condition (a) and the invariant N of the forest of $(\mathcal{I}_1)_\tau$ has dropped, we may use induction to finish the proof.

(b) *implies* (a). It is straightforward to reduce it to the case where in our family the parameter space T is an integral scheme, with generic point τ and function field K. We may also assume that all the points in $V(I_\tau)$ are K-rational. In fact, as in the proof of Lemma (2.3), all points of $V(I_\tau)$ are defined over a finite extension K' of K; taking $T' =$ integral closure of T in K' (which is again a noetherian scheme, finite over T, by the excellence assumption), it suffices to prove the theorem for the induced family over T', which satisfies the stated rationality condition.

So, assuming that $(Z \xrightarrow{\pi} T, I)$ satisfies these extra conditions, let

$$V(I) = V_1 \cup \cdots \cup V_s \cup W,$$

where the V_i's are the irreducible components of $V(I)$ such that $\pi(V_i) = T$ while W is the union of the other irreducible components.

We may assume, after a finite surjective base change if necessary, that π induces an isomorphism $\pi_i\colon V_i \xrightarrow{\sim} T$, for all $i = 1, \ldots, s$. In fact, if π_1 is not an isomorphism, consider $\pi_1\colon V_1 \to T$ and the family (π', I') (parametrized by V_1) induced by (π, I). By elementary considerations, we get a new expression $V(I') = V_1' \cup \cdots \cup V_s' \cup W'$ analogous to the previous one, and a section s_1 such that $s_1(T) = V_1'$. If the induced projection $V_2' \to T$ is not an isomorphism, continue in a similar way with V_2', and so on.

Next, we claim that W must be empty. We'll check this by contradiction. First of all, by Theorem (4.1) and the H−D formula (see the proof of (3.2)), condition (b) implies that the function $e\colon T \to \mathbf{Z}$ of (4.1) must be constant. Let J be the largest Z-ideal such that $J_\tau = I_\tau$. Clearly, $V(J) = V_1 \cup \cdots \cup V_s$. Were $W \neq \emptyset$, then by Theorem (4.1)(i), for $t \in \pi(W)$ we'd get $e(t, J) < e(t, I)$. Then, we have

$$e(\tau, I) = e(\tau, J) \leq e(t, J) < e(t, I) = e(\tau, I)$$

(the first equality because $I_\tau = J_\tau$, the second by Theorem (4.1)(ii)). This is a contradiction.

Finally, note that $V_i \cap V_j = \emptyset$ if $i \neq j$. Otherwise, if $z \in V_i \cap V_j$ and $t = \pi(z)$, then the forest of $I_{\bar{t}}$ would have fewer vertices of level 0 than that of $I_{\bar{\tau}}$, contradicting (b). In other words, A (the Z-ideal defining $V(I)$), is an ideal of sections of π.

Blow-up Z along A, to get $Z_1 \to Z$ and let I_1 the the proper transform of I to Z_1. Then, by (4.4), (Z_1, I_1) is a new family clearly satisfying (b). By induction on the height $N(\gamma)$ (with γ the forest corresponding to τ) we have a finite surjective base change $T' \to T$ (where we may assume T' again integral) such that if $Z_1' = Z_1 \times_T T'$, the induced family $(Z_1', I_1 O_{Z_1'})$ is resolvable by sections. Since, by Proposition (4.3), Z_1' can be identified to the blowing-up of $Z' := Z \times_T T'$ along $I O_{Z'}$, we see that $(Z', I O_{Z'})$ can be resolved by sections, i.e., that (a) holds. ∎

COROLLARY (4.8). *Let* $(Z \xrightarrow{\pi} T, \mathcal{I})$ *be a family of ideals, with* T *integral. Then, there is a non-empty open set* $U \subseteq T$ *such that the equivalent conditions* (a) *and* (b) *hold for the family induced over* U.

Proof. Let γ be the forest at the geometric generic point, $N(\gamma)$ its height. If $N(\gamma) = -1$, i.e., $V(\mathcal{I}_{\bar{\tau}}) = \emptyset$, then the assertion holds for $U = T - \pi(V(\mathcal{I}))$. So assume that $N(\gamma) \geq 0$, and let s be the number of vertices of level 0. In the proof of (b) ⇒ (a) (Theorem (4.7)) we saw that, after a finite, surjective base change (if necessary), we may assume that the points of $V(\mathcal{I}_\tau)$ are rational over $k(\tau)$, and that we have an expression

$$V(\mathcal{I}) = V_1 \cup \cdots \cup V_s \cup W,$$

where the V_i's are the irreducible components of $V(\mathcal{I})$ such that $\pi(V_i) = T$, W is the union of the other irreducible components, and the projection induces isomorphisms $\pi_i: V_i \xrightarrow{\sim} T$. Set $C = W \cup (\bigcup_{i,j}(V_i \cap V_j))$. Then clearly $C_\tau = \emptyset$ and $\mathcal{A}'$, the restriction to $U' = T - \pi(C)$ of $\mathcal{A}$ (the Z-ideal of $V(\mathcal{I})$) is an ideal of sections. Restrict the family over U' and blow-up along $\mathcal{A}'$. As in the proof of Theorem (4.7) we may proceed by induction to N, to find a base change $U'' \to U'$, such that the induced family satisfies (a) (or, equivalently, (b)). The image of the composition $U'' \to T$ (a dominant map) contains an open set U of T, which satisfies all the requirements. ∎

We leave to the reader the task of checking the additional details (involving proximities and exceptional divisors) necessary to obtain, along very similar lines, the following results (cf. [NV2]).

THEOREM (4.9). *Let* $(\pi, \mathcal{I})$ *be as in Definition* (4.5), *with* T *connected. Then, this family satisfies condition* (a′) *if and only if it satisfies condition* (b′).

COROLLARY (4.10). *Let* $(Z \xrightarrow{\pi} T, \mathcal{I})$ *be a family of ideals, with* T *integral. Then, there is a non-empty open set* $U_0 \subseteq T$ *such that the equivalent conditions* (a′) *and* (b′) *hold for the family induced over* U.

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